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QED_{2+1} with Nonzero Fermion Density and Quantum Hall Effect

Vadim Zeitlin[†]

Department of Theoretical Physics,
P. N. Lebedev Physical Institute,
Leninsky prospect 53, 117924 Moscow, Russia

Abstract

A general expression for the conductivity in the QED_{2+1} with nonzero fermion density in the uniform magnetic field is derived. It is shown that the conductivity is entirely determined by the Chern-Simons coefficient: $\sigma_{ij} = \varepsilon_{ij} \mathcal{C}$ and is a step-function of the chemical potential and the magnetic field.

[†] E-mail address: zeitlin@lpi.ac.ru

We shall present here a simple relativistic model possessing a quantum Hall-like conductivity – (Maxwell) quantum electrodynamics on the plane with nonzero fermion density. Using the general properties of the QED₂₊₁ an expression for the conductivity will be derived. In this model a transverse conductivity arises owing to the induced Chern-Simons term, which is generated dynamically in the one-loop polarization operator. The quantization of the Chern-Simons coefficient at a nonzero chemical potential and magnetic field manifests itself in the quantization of the conductivity. The latter is a step-function of the number of the filled Landau levels.

We shall consider (2+1)-dimensional QED with a nonzero chemical potential. Its Lagrangian reads

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\imath\rlap{\not{D}} + e\rlap{\not{A}} + \gamma_0\mu - m)\psi \quad (1)$$

with standard notations [1]: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the magnetic field B is defined as $B = \partial_1 A_2 - \partial_2 A_1$, γ -matrices are Pauli matrices, $\gamma_0 = \sigma_3$, $\gamma^{1,2} = \imath\sigma_{1,2}$, μ is the chemical potential (by introducing the chemical potential in this way one has to modify the $\imath\epsilon$ -prescription in the fermion Green function [2, 3]). In this model the fermion mass term violates \mathcal{P} - and \mathcal{T} -parity, thus the Chern-Simons term may be generated dynamically despite the fact it is not present in the bare Lagrangian [1].

In a strong background magnetic field the current \tilde{I} induced by the perturbing electric field may be written as a linear response function:

$$\tilde{I}_\mu(x) = \int d^3x' \Pi_{\mu\nu}(x - x'|B, \mu) \tilde{A}^\nu(x') \quad , \quad (2)$$

$\Pi_{\mu\nu}$ is the polarization operator, $\partial_i \tilde{A}^0(x) = E_i$, $\tilde{A}_i(x) = 0$.

Using the definition of the conductivity $\sigma_{ij} = \left. \frac{\partial I_i}{\partial E_j} \right|_{E \rightarrow 0}$ one may rewrite Eq. (2) as follows:

$$\sigma_{ij} = \imath \left. \frac{\partial \Pi_{0i}(p)}{\partial p_j} \right|_{p \rightarrow 0} . \quad (3)$$

Therefore, the calculation of the conductivity is reduced to the calculation of the polarization operator in the QED₂₊₁ with nonzero fermion density with some uniform magnetic field. Before the calculation of the polarization operator it is worth to obtain its general tensor structure. As it is shown in Appendix, the polarization operator may be written in the following form:

$$\Pi_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \mathcal{A} + \left(\frac{p_\mu p_\nu}{p^2} - \frac{p_\mu u_\nu + u_\mu p_\nu}{(pu)} + \frac{u_\mu u_\nu}{(pu)^2} p^2 \right) \mathcal{B} + \imath \varepsilon_{\mu\nu\alpha} p^\alpha \mathcal{C} . \quad (4)$$

The first tensor structure in Eq. (4) is the standard vacuum term, the second one is usually associated to the finite-temperature effects [4] and the last one is the induced Chern-Simons term [1].

Since the scalars \mathcal{A} , \mathcal{B} and \mathcal{C} are the functions of μ , B , p_0 and \mathbf{p}^2 and $\Pi_{\mu\nu}$ is finite in the $p \rightarrow 0$ limit, the only term that may survive in the expression for the conductivity, Eq. (3), is the Chern-Simons term

$$\sigma_{ij} = \varepsilon_{ij}\sigma = \varepsilon_{ij}\mathcal{C} \quad (3')$$

and our task is reduced to the calculation of the Chern-Simons coefficient in the limit $p_0 = 0$, $\mathbf{p}^2 \rightarrow 0$. From the other hand, it is well-known that a nonzero contribution to the Chern-Simons coefficient \mathcal{C} in the $p \rightarrow 0$ limit arises on the one-loop level only [5].

Now the problem is sufficiently simplified: to get the whole answer for the conductivity we may calculate one-loop Chern-Simons term only. It is possible to calculate this directly¹ since the one-loop polarization operator may be written as

$$\Pi_{\mu\nu}(p|\mu, B) = ie^2 \text{tr} \int d^3q \gamma_\mu G(p+q|\mu, B) \gamma_\nu G(q|\mu, B) \quad (5)$$

and the corresponding expression for the fermion Green function $G(p|\mu, B)$ is known [8]. At the same time, $\Pi_{\mu\nu}(p)$ is:

$$\Pi_{\mu\nu}(x, x') = i \frac{\delta \langle j_\mu(x) \rangle}{\delta A_\nu(x')} \quad (6)$$

and its components Π_{0j} , ($j = 1, 2$) in the static limit are:

$$\Pi_{0j}(p \rightarrow 0) = i e \varepsilon_{ij} p_i \frac{\partial \rho}{\partial B} \quad , \quad (6')$$

ρ is the fermion density, $j_o = e\rho$.

The simplest way to calculate the fermion density in the QED₂₊₁ with magnetic field is based on the index theorem [9]: the fermion number N is proportional to a difference between the number of positive and negative energy levels. At $T = 0$, $\mu \neq 0$ the fermion number N is:

$$N = -\frac{1}{2} \sum_k \text{sign}(\varepsilon_k) + \sum_k (\theta(\varepsilon_k) \theta(\mu - \varepsilon_k) - \theta(-\varepsilon_k) \theta(\varepsilon_k - \mu)) \quad , \quad (7)$$

¹In Ref. [6] the one-loop polarization operator for the QED₂₊₁ with the uniform external field was calculated directly for the particular choices of the chemical potential corresponding to the vacuum and the filled lowest Landau level. The results for the Chern-Simons coefficient for these cases coincide with those obtained here and in Ref. [7]. There is a discrepancy in the calculation of the coefficient \mathcal{B} in Ref. [6] and Ref. [7] which does not affect the results presented here.

ε_k are the energy levels. In the QED₂₊₁ with a background uniform magnetic field B the fermion energy spectrum (Landau levels) is discrete,

$$p_0 = -m \operatorname{sign}(eB), \quad p_0 = \pm \sqrt{m^2 + 2|eB|n}, \quad n = 1, 2, 3, \dots \quad (8)$$

(note the asymmetry of the spectrum). Using Eq. (7) and taking into account the degeneracy of the Landau levels $\frac{|eB|}{2\pi}$, the fermion density may be written as follows ($eB > 0$):

$$\rho(B, \mu) = \frac{eB}{4\pi} + \begin{cases} \frac{eB}{2\pi} \left[\frac{\mu^2 - m^2}{2eB} \right], & \mu > m; \\ 0, & |\mu| < m; \\ -\frac{eB}{2\pi} \left(1 + \left[\frac{\mu^2 - m^2}{2eB} \right] \right), & \mu < -m, \end{cases} \quad (9)$$

where $[\dots]$ denotes the integral part.

Now the conductivity may be written as:

$$\sigma = \frac{e^2}{4\pi} + \begin{cases} \frac{e^2}{2\pi} \left[\frac{\mu^2 - m^2}{2eB} \right], & \mu > m; \\ 0, & |\mu| < m; \\ -\frac{e^2}{2\pi} \left(1 + \left[\frac{\mu^2 - m^2}{2eB} \right] \right), & \mu < -m. \end{cases} \quad (10)$$

It is easy to see that the conductivity σ as a function of μ at a constant magnetic field or as a function of B at a fixed chemical potential is a step-function, therefore we have the integer quantum Hall effect. At the same time, for these two cases the jumps of the conductivity are accompanied by sharp changes of density, thus we have a "naive" quantum Hall effect.

From the other hand we may consider an equation

$$\rho(B, \mu) = \bar{\rho} \quad , \quad (11)$$

$\bar{\rho}$ is constant. This equation has an infinite set of solutions, namely

$$eB_n = \frac{4\pi\bar{\rho}}{2n+1}, \quad n = 0, 1, 2, \dots, \quad (12)$$

see Figure 1. The conductivity of these states (of equal density) is quantized, $\sigma_n \sim (2n+1)$.

In all above-described schemes step-like variations of the conductivity are connected to filling (emptying) of the successive Landau levels: a discrete fermion spectrum (as well as equal degeneracy of Landau levels) leads to the quantization of the conductivity in the QED₂₊₁.

The procedure described may be applied also for the calculation of the conductivity in the QED₃₊₁ in the plane orthogonal to the direction of magnetic field. At zero temperature fermion density in QED₃₊₁ is a continuous (but not smooth) function of B and μ [10]. This leads to the relativistic Schubnikov - de Haas oscillations of conductivity.

In conclusion, we want to stress that approach used in this paper for the calculation of the conductivity, Eqs. (3), (6') is valid for nonzero electric field, too, but the calculation of the fermion density for μ , B and $E \neq 0$ is more complicate .

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Appendix

To find a general tensor structure of the polarization operator $\Pi_{\mu\nu}(p)$ in the QED₂₊₁ we shall write its eigenvector decomposition (see, e.g. [11]):

$$\Pi_{\mu\nu}(p) = \sum_{i=1,2} \kappa_i \frac{b_\mu^{(i)} b_\nu^{(i)*}}{b_\alpha^{(i)} b^{\alpha(i)*}} , \quad (A1)$$

the third eigenvector with zero eigenvalue is p_μ (due to the gauge invariance $\Pi_{\mu\nu}(p)$ is transversal, $\Pi_{\mu\nu}(p) p^\nu = 0$).

The eigenvectors $b^{(i)}$ may be chosen as follows:

$$b_\mu^{(i)} = \varepsilon_{\mu\alpha\beta} u^\alpha p^\beta + i\alpha_i (p_\mu - u_\mu \frac{p^2}{pu}), \quad b_\mu^{(i)} p^\mu \equiv 0 , \quad (A2)$$

u^μ is the 3-velocity of the medium [4], $u^\mu = (1, 0, 0)$.

The orthogonality condition $b_\alpha^{(1)} b^{\alpha(2)*} = 0$ fixes one of the coefficients α_i , $\alpha_1 \alpha_2^* = -\frac{(pu)^2}{p^2}$ and Eq. (A1) may be rewritten as follows:

$$\begin{aligned}
\Pi_{\mu\nu}(p) &= \frac{(\kappa_1 + \kappa_2 \lambda)}{(1 + \lambda)(p^2 - (pu)^2)} \varepsilon_{\mu\alpha\beta} u^\alpha p^\beta \varepsilon_{\nu\xi\phi} u^\xi p^\phi \\
&+ \frac{(\kappa_1 \lambda + \kappa_2)}{(1 + \lambda)(p^2 - (pu)^2)} \frac{(pu)^2}{p^2} (p_\mu - u_\mu \frac{p^2}{(pu)}) (p_\nu - u_\nu \frac{p^2}{(pu)}) \\
&+ i \Re e \alpha_{(1)} \cdot \frac{(\kappa_1 - \kappa_2)}{(1 + \lambda)(p^2 - (pu)^2)} \left(\varepsilon_{\mu\alpha\beta} u^\alpha p^\beta (p_\nu - u_\nu \frac{p^2}{(pu)}) - (p_\mu - u_\mu \frac{p^2}{(pu)}) \varepsilon_{\nu\xi\phi} u^\xi p^\phi \right) \\
&- \Im m \alpha_{(1)} \cdot \frac{(\kappa_1 - \kappa_2)}{(1 + \lambda)(p^2 - (pu)^2)} \left(\varepsilon_{\mu\alpha\beta} u^\alpha p^\beta (p_\nu - u_\nu \frac{p^2}{(pu)}) + (p_\mu - u_\mu \frac{p^2}{(pu)}) \varepsilon_{\nu\xi\phi} u^\xi p^\phi \right)
\end{aligned} \tag{A3}$$

$$\lambda = |\alpha_1|^2 \frac{p^2}{(pu)^2}.$$

The last term in Eq. (A3) violates \mathcal{PT} -parity, therefore the condition $\Im m \alpha_{(1)} = 0$ must hold and $\Pi_{\mu\nu}(p)$ may be decomposed over the three tensor structures (cf. [12]). After exclusion of the last term in Eq. (A3) the polarization operator may be presented in a more convenient way:

$$\begin{aligned}
\Pi_{\mu\nu}(p) &= \frac{\kappa_1 + \lambda \kappa_2}{\lambda + 1} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + i \frac{\lambda^{1/2}}{\lambda + 1} (\kappa_1 - \kappa_2) \varepsilon_{\mu\nu\alpha} p^\alpha / p \\
&+ \frac{\kappa_1 - \kappa_2}{\lambda + 1} \frac{\lambda - 1}{p^2 / (pu)^2 - 1} \left(\frac{p_\mu p_\nu}{p^2} - \frac{p_\mu u_\nu + u_\mu p_\nu}{(pu)} + \frac{u_\mu u_\nu}{(pu)^2} p^2 \right) .
\end{aligned} \tag{A4}$$

References

- [1] S. Deser, R. Jackiw, and S. Templeton, *Ann. Phys.* **140** (1982) 372.
- [2] E. V. Shuryak, *Phys.Rep.* **61** (1980) 73.
- [3] A. Chodos, K. Everdin, and D. Owen, *Phys. Rev.* **D42** (1990) 2881.
- [4] E. S. Fradkin, *Proceedings of P. N. Lebedev Physical Institute* **29** (1965) 7.
- [5] T. Bernstein and A. Lee, *Phys. Rev.* **D32** (1985) 1020;
S. Coleman and B. Hill, *Phys.Lett.* **B159** (1985) 184;
Y. Nagahami, *Z. Phys.* **C37** (1986) 583.
- [6] Y. Hosotani, *Phys.Lett.* **B319** (1993) 332.
- [7] Vad. Zeitlin, *Sov.J.Nucl.Phys.* **49** (1989) 712.
- [8] Vad. Zeitlin, *Mod. Phys. Lett.* **A8** (1993) 1821.
- [9] A. Niemi, *Nucl.Phys.* **B251** (1985) 155.
- [10] D. Persson and Vad. Zeitlin, preprint (FIAN/TD/94-01, Göteborg ITP 94-11, hep-ph/9404216), to appear in *Phys. Rev.* **D**.
- [11] A. E. Shabad, *Ann. Phys.* **90** (1975) 166.
- [12] R. Randjbar-Daemi, A. Salam, and J. Strathdee, *Nucl. Phys.* **B340** (1990) 403.